LOOP HOMOLOGY OF QUATERNIONIC PROJECTIVE SPACES

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Abstract. We determine the Batalin-Vilkovisky algebra structure of the integral loop homology of quaternionic projective spaces and octonionic projective plane.

1. Introduction

Let M be a closed oriented manifold of dimension d and let $LM = \mathrm{Map}(S^1, M)$ denote its free loop space. By loop homology we understand the homology groups of LM with the degree shifted by $-d$

$$\mathbb{H}_*(LM) = H_{*+d}(LM).$$

In [2] it was shown that this graded group can be equipped with a product and an operator Δ giving $\mathbb{H}_*(LM)$ the structure of a Batalin-Vilkovisky algebra. The methods computing the product on concrete manifolds are based either on the modified Serre spectral sequence derived in [4] or on the isomorphism of the loop homology of M with the Hochschild cohomology $HH^*(C^*(M); C^*(M))$ of the cochain complex as rings, [3]. There is also a way of defining a BV-structure on $HH^*(C^*(M); C^*(M))$, [11]. The BV-algebra structures on the loop homology and the Hochschild cohomology are isomorphic over the fields of characteristic zero ([5]) but not over other coefficients in general. Hence the computation of the BV operator is more subtle. So far the BV-algebra structure of the loop homology with integral coefficients has been determined for the Lie groups [7], for the spheres [9], for the complex Stiefel manifolds [10] and for the complex projective spaces [8]. Over rationals it has been described for the quaternionic projective spaces [13] and the surfaces [12].

The aim of this note is to describe the BV-algebra structure of the integral loop homology of the quaternionic projective spaces $\mathbb{H}P^n$ and the octonionic projective plane $\mathbb{O}P^2$.

Theorem 1.1. *The string topology BV-algebra structure of $\mathbb{H}P^n$ is given by*

$$\mathbb{H}_*(L\mathbb{H}P^n; \mathbb{Z}) \cong \frac{\mathbb{Z}[a, b, x]}{\langle a^{n+1}, b^2, a^n \cdot b, (n+1)a^n \cdot x \rangle}$$

with $a \in \mathbb{H}_{-4}(L\mathbb{H}P^n; \mathbb{Z})$, $b \in \mathbb{H}_{-1}(L\mathbb{H}P^n; \mathbb{Z})$ and $x \in \mathbb{H}_{4n+2}(L\mathbb{H}P^n)$, and

$$\Delta(a^p x^q) = 0, \quad \Delta(a^p b x^q) = [(n-p) + q(n+1)]a^p x^q$$

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for all $0 \le p \le n$, $0 \le q$.

Let us note that for $n = 1$ the quaternionic projective space is S^4 and the statement agrees with the result obtain by L. Menichi in [9] for even dimensional spheres.

Theorem 1.2. *There are elements* $a \in \mathbb{H}_{-8}(L\mathbb{O}P^2;\mathbb{Z})$, $b \in \mathbb{H}_{-1}(L\mathbb{O}P^2;\mathbb{Z})$ *and* $x \in \mathbb{H}_{22}(L\mathbb{O}P^2)$ *such that the string topology BV-algebra structure of* $\mathbb{O}P^2$ *is given by*

$$\mathbb{H}_*(L\mathbb{O}P^2;\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,x]}{\langle a^3, b^2, a^2\cdot b, 3a^n\cdot x\rangle}$$

and

$$\Delta(a^p x^q) = 0, \quad \Delta(a^p b x^q) = (2+3q-p)a^p x^q$$

for all $0 \le p \le 2$, $0 \le q$.

The statements of Theorem 1.1 and 1.2 concerning the ring structure are consequences of the computation of $HH^*(\mathbb{Z}[y]/y^{n+1};\mathbb{Z}[y]/y^{n+1})$ in [13] and the ring isomorhism between the loop homology and the Hochschild cohomology. Nevertheless, we provide an alternative proof using the Serre spectral sequence for the fibrations $\Omega M \to LM \to M$ converging to the ring $\mathbb{H}_*(LM;\mathbb{Z})$. These computations will be carried out in the next section.

In the last section we will show what the BV operator Δ looks like. We use the knowledge of Δ on S^4 and S^8 and the inclusions $S^4 = \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^n$ and $S^8 \hookrightarrow \mathbb{O}P^2$. The computation will be completed by comparing Δ in integral homology with BV-operator Δ in rational cohomology computed by Yang in [13]. The results show that for the quaternionic projective spaces and the octonionic projective plane the BV-algebra structures on the loop homology and the Hochschild homology over integers are isomorphic (in contrast to the complex projective spaces, see [8]).

2. The ring structure of loop homology

According to [4] the spectral sequence for the fibration $\Omega M \to LM \to M$ with $E^2_{p,q} = H^{-p}(M; H_q(\Omega M;\mathbb{Z}))$ and the product coming from the Pontryagin product in $H_*(\Omega M;\mathbb{Z})$ and the cup product in $H^*(M; H_*(\Omega M;\mathbb{Z})$ converges to $\mathbb{H}_{p+q}(LM;\mathbb{Z})$ as an algebra. To apply this spectral sequence to $M = \mathbb{H}P^n$ we have to determine the Pontryagin ring $H_*(\Omega\mathbb{H}P^n;\mathbb{Z})$. We will consider $n \ge 2$ since for $\mathbb{H}P^1 = S^4$ the statement of Theorem 1.1 has been proved in [9].

Lemma 2.1. *For* $n \ge 2$ *the Pontryagin ring structure of* $H_*(\Omega\mathbb{H}P^n;\mathbb{Z})$ *is given by*

$$H_*(\Omega\mathbb{H}P^n;\mathbb{Z}) \cong \mathbb{Z}[x] \otimes \Lambda[t]$$

where the degrees of x *and* t *are* $4n+2$ *and* 3, *respectively.*

Proof. The Hopf fibration $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ gives us the fibration

$$\Omega S^{4n+3} \xrightarrow{j} \Omega\mathbb{H}P^n \xrightarrow{p} S^3 \tag{2.1}$$

Since $p_* : \pi_k(\Omega\mathbb{H}P^n) \to \pi_k(S^3)$ is an isomorphism for $0 \le k \le 6$, there is up to homotopy a unique map $i : S^3 \to \Omega\mathbb{H}P^n$ such that $p \circ i$ is homotopic to the identity

on S^3. Therefore the long exact sequence of homotopy groups for this fibration passes to short exact sequences which split:

$$0 \longrightarrow \pi_*(\Omega S^{4n+3}) \xrightarrow{j_*} \pi_*(\Omega \mathbb{H}P^n) \underset{i_*}{\overset{p_*}{\rightleftarrows}} \pi_*(S^3) \longrightarrow 0$$

Denote by μ the Pontryagin product on $\Omega\mathbb{H}P^n$. The map $h = \mu \circ (j,i) : \Omega S^{4n+2} \times S^3 \to \Omega\mathbb{H}P^n$ is a homotopy equivalence since it induces an isomorphism of homotopy groups. So we obtain an isomorphism of homology groups

$$H_*(\Omega\mathbb{H}P^n;\mathbb{Z}) \cong H_*(\Omega S^{4n+3};\mathbb{Z}) \otimes H_*(S^3;\mathbb{Z}) \cong \mathbb{Z}[x] \otimes \Lambda[t].$$

The Pontryagin ring structure of $H_*(\Omega\mathbb{H}P^n;\mathbb{Z})$ can be recovered using the duality between the Hopf algebras $H_*(\Omega\mathbb{H}P^n;\mathbb{Z})$ and $H^*(\Omega\mathbb{H}P^n;\mathbb{Z})$. The map h induces an algebra isomorphism $h^* : H^*(\Omega\mathbb{H}P^n;\mathbb{Z}) \to H^*(\Omega S^{4n+3};\mathbb{Z}) \otimes H^*(S^3;\mathbb{Z})$. We know that $H^*(\Omega\mathbb{H}P^n;\mathbb{Z})$ is a commutative associative Hopf algebra with μ^* as a coproduct. As an algebra $H^*(\Omega\mathbb{H}P^n;\mathbb{Z}) \cong \Gamma_{\mathbb{Z}}[\alpha_1,\alpha_2,\dots] \otimes \Lambda[\beta]$, where $\Gamma_{\mathbb{Z}}[\alpha_1,\alpha_2,\dots]$ is a divided polynomial algebra with generators α_i and relations $\alpha_i\alpha_j = \binom{i+j}{i}\alpha_{i+j}$. Since $j^* : H^*(\Omega\mathbb{H}P^n;\mathbb{Z}) \to H^*(\Omega S^{4n+3};\mathbb{Z})$ is a homomorphism of Hopf algebras and the Hopf algebra structure of $H^*(\Omega S^{4n+3};\mathbb{Z})$ is well known, the coproduct on $H_*(\Omega\mathbb{H}P^n;\mathbb{Z})$ is given by

$$\mu^*(\beta) = \beta \otimes 1 + 1 \otimes \beta, \quad \mu^*(\alpha_k) = \sum_{k=i+j} \alpha_i \otimes \alpha_j,$$
$$\mu^*(\alpha_k\beta) = \sum_{k=i+j} \alpha_i\beta \otimes \alpha_j + \sum_{k=i+j} \alpha_i \otimes \beta\alpha_j.$$

By duality this coproduct completely determines the Potryagin product in $H^*(\Omega\mathbb{H}P^n;\mathbb{Z})$. Let $t \in H_*(\Omega\mathbb{H}P^n)$ be a dual element to β, x_k be a dual to α_k and z_k be a dual to $\alpha_k\beta$. Then

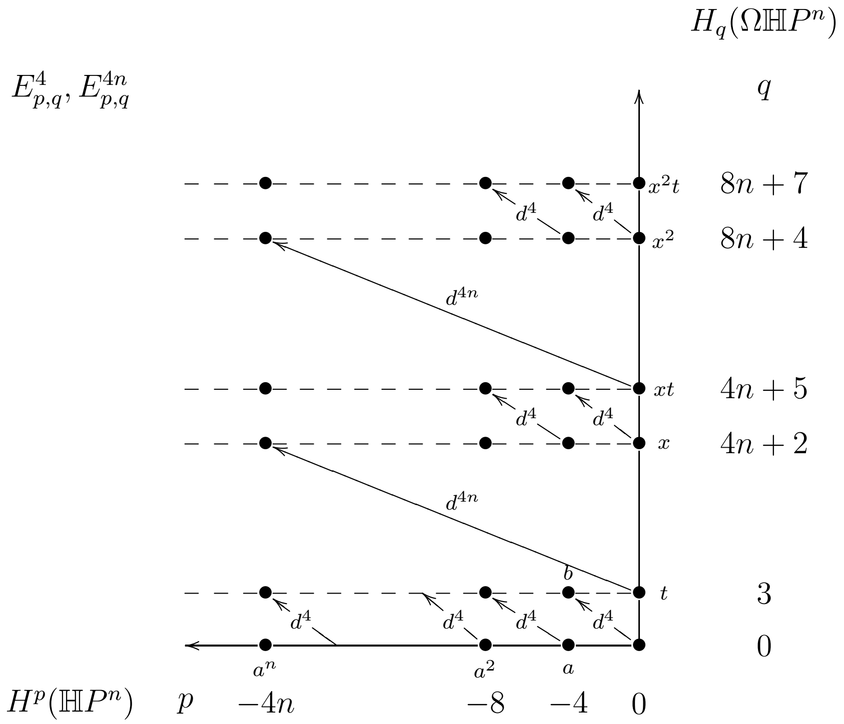
$$x_{i+j} = x_i x_j, \quad z_{i+j} = z_i x_j.$$

If we put $x = x_1$, we obtain $x_i = x^i$ and $z_i = x^i t$ for all $i \geq 0$. This completes the proof. $\square$

Now we return to the spectral sequence converging to the algebra $\mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Z})$. Its E^2 term is

$$E^2_{-p,q} = H^p(\mathbb{H}P^n; H_q(\Omega\mathbb{H}P^n;\mathbb{Z})) \cong H^p(\mathbb{H}P^n;\mathbb{Z}) \otimes H_q(\Omega\mathbb{H}P^n;\mathbb{Z}) \cong \frac{\mathbb{Z}[a] \otimes \mathbb{Z}[x,t]}{\langle a^{n+1}, t^2\rangle}$$

where $a \in H^4(M;\mathbb{Z})$ and x, t as in Lemma 2.1. The stages E^4 and E^{4n} of the spectral sequence with possible nonzero differentials are shown in the following diagram:



Since $E^\infty_{p,q} \Rightarrow \mathbb{H}_{p+q}(L\mathbb{H}P^n;\mathbb{Z}) = H_{p+q+4n}(L\mathbb{H}P^n;\mathbb{Z})$ we can determine the differentials from the knowledge of the additive structure of $H_*(L\mathbb{H}P^n;\mathbb{Z})$.

To compute it we use the result of [1] on the existence of a stable decomposition

$$(L\mathbb{H}P^n)_+ \simeq \mathbb{H}P^n_+ \vee \bigvee_{l\geq 1} S(\eta)^{l\xi\oplus(l-1)\zeta}$$

where η is tangent bundle of the quaternionic projective space $\mathbb{H}P^n$, ξ is the 3-dimensional Lie algebra bundle over $\mathbb{H}P^n$ and ζ is the fibrewise tangent bundle of $S(\eta)$ and $S(\eta)^\omega$ stands for the Thom space of the vector bundle ω over $S(\eta)$. Note that $\dim S(\eta) = 8n-1$ and $\dim\zeta = 4n-1$. Using the Gysin long exact sequence for the fibration $S^{4n-1} \to S(\eta) \to \mathbb{H}P^n$ and the fact that the Euler characterictic class of η is an $(n+1)$-multiple of the generator $a^n \in H^{4n}(\mathbb{H}P^n;\mathbb{Z})$ we get

$$H_iS(\eta) = \begin{cases} \mathbb{Z} & i = 0, 4, \dots, 4n-4, 4n+3, 4n+7, \dots, 8n-1, \\ \mathbb{Z}_{n+1} & i = 4n-1, \\ 0 & \text{otherwise.} \end{cases}$$

The dimesion of the vector bundle $l\xi \oplus (l-1)\zeta$ is $4n(l-1)+2l+1$, so due to the Thom isomorphism

$$H_*(L\mathbb{H}P^n;\mathbb{Z}) \cong H_*(\mathbb{H}P^n;\mathbb{Z}) \oplus \bigoplus_{l\geq 1} H_{*+4n(l-1)+2l+1}(S(\eta);\mathbb{Z}).$$

Since $\mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Z}) \cong E^\infty_{*,*}$, the E^∞ stage of the spectral sequence is the following

$$
\begin{array}{cccccccc}
 & & & & & & \uparrow & q \\
- - - & \mathbb{Z} - - - & \mathbb{Z} - - - - - - & \mathbb{Z} - - - & \mathbb{Z} - - - & & 0 & 8n+7 \\
- \quad - & \mathbb{Z}_{n+1} - \quad - & \mathbb{Z} - - - - - - & \mathbb{Z} - - - & \mathbb{Z} - \quad - & & \mathbb{Z} & 8n+4 \\
 & & & & & & | & \\
- - - & \mathbb{Z} - - - & \mathbb{Z} - - - - - - & \mathbb{Z} - - - & \mathbb{Z} - - - & & 0 & 4n+5 \\
- \quad - & \mathbb{Z}_{n+1} - \quad - & \mathbb{Z} - - - - - - & \mathbb{Z} - - - & \mathbb{Z} - \quad - & & \mathbb{Z} & 4n+2 \\
 & & & & & & | & \\
- - - & \mathbb{Z} - - - & \mathbb{Z} - - - - - - & \mathbb{Z} - - - & \mathbb{Z} - - - & & 0 & 3 \\
\longleftarrow & \mathbb{Z} \longrightarrow & \mathbb{Z} \longrightarrow & \mathbb{Z} \longrightarrow & \mathbb{Z} \longrightarrow & & \mathbb{Z} & 0 \\
p & -4n & & -8 & -4 & & 0 &
\end{array}
$$

It forces the differentials d^4 in E^4 to be zero and the differentials $d^{4n} : E^{4n}_{0,(4n+2)i+3} \to E^4_{-4n,(4n+2)(i+1)}$ to be the multiplication by $n+1$.

So $E^\infty_{*,*}$ as a ring is generated by the group generators $a \in E^\infty_{-4,0} \cong H^4(\mathbb{H}P^n;\mathbb{Z})$, $x \in E^\infty_{0,4n+2} \cong H_{4n+2}(\Omega\mathbb{H}P^n;\mathbb{Z})$ and $b \in E^\infty_{-4,3} \cong H^4(\mathbb{H}P^n;\mathbb{Z}) \otimes H_3(\Omega\mathbb{H}P^n;\mathbb{Z})$ which satisfy relations $a^{n+1} = 0$, $(n+1)\, x \otimes a^n = 0$, $b \otimes a^n = 0$, $b^2 = 0$. We conclude that as rings

$$\mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Z}) \cong E^\infty_{*,*} \cong \frac{\mathbb{Z}[a,b,x]}{\langle a^{n+1}, b^2, a^n b, (n+1)a^n x\rangle}.$$

In the case of the octonionic projective plane the derivation of the ring structure of the loop homology follows the same lines.

Lemma 2.2. *The Pontryagin ring structure of $H_*(\Omega\mathbb{O}P^2;\mathbb{Z})$ is given by*

$$H_*(\Omega\mathbb{O}P^2;\mathbb{Z}) \cong \mathbb{Z}[x] \otimes \Lambda[t]$$

where $|x| = 22$ and $|t| = 7$.

Proof. Using the fact that

$$H^*(\Omega\mathbb{O}P^2) \cong \Gamma_{\mathbb{Z}}[\alpha_1, \alpha_2, \dots] \otimes \Lambda[\beta]$$

where $|\alpha_i| = 22i$ and $|\beta| = 7$, proved in [6], we can proceed in the same way as in the proof of Lemma 2.1. □

The additive structure of $H_*(L\mathbb{O}P^2;\mathbb{Z})$ was found in [1] using a stable decomposition of $L\mathbb{O}P^2$ derived there:

$$H_i(L\mathbb{O}P^2) = \begin{cases} \mathbb{Z} & i = 0, 8, 16, 22m-15, 22m-7, 22m+8, 22m+16, \\ \mathbb{Z}_3 & i = 22m, \\ 0 & \text{otherwise.} \end{cases}$$

It yields that in the spectral sequence starting with

$$E^2_{-p,q} = H^p(\mathbb{O}P^2; H_q(\Omega\mathbb{O}P^2;\mathbb{Z})) \cong H^p(\mathbb{O}P^2;\mathbb{Z}) \otimes H_q(\Omega\mathbb{O}P^2;\mathbb{Z}) \cong \frac{\mathbb{Z}[a]\otimes\mathbb{Z}[x,b]}{\langle a^3, b^2\rangle}$$

all the differentials are zero with the exception of the differentials $d^{16}: E^{16}_{0,22m-15} \to E^{16}_{-16,22m}$ which act as the multiplication by 3. The group generators $a \in E^\infty_{-8,0} \cong H^*(\mathbb{O}P^2;\mathbb{Z})$, $x \in E^\infty_{0,22} \cong H_{22}(\Omega\mathbb{O}P^2;\mathbb{Z})$ and $b \in E^\infty_{-8,7} \cong H^8(\mathbb{H}P^n;\mathbb{Z}) \otimes H_7(\Omega\mathbb{O}P^2;\mathbb{Z})$, generate $E^\infty_{*,*} \cong \mathbb{H}_*(L\mathbb{O}P^2;\mathbb{Z})$ as a ring satisfying relations $a^3 = 0$, $b^2 = 0$, $3ax = 0$ and $a^2b = 0$.

3. The BV operator

The BV operator $\Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$ and its unshifted version $\Delta' : H_*(LM) \to H_{*+1}(LM)$ come from the canonical action of S^1 on LM. So any map $f : N \to M$ between manifolds induces a homomorhism $H_*(LN) \to H_*(LM)$ which commutes with Δ'. To determine the BV operator on $\mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Z})$ and $\mathbb{H}_*(L\mathbb{O}P^2;\mathbb{Z})$ we use this fact for the inclusions $S^4 \hookrightarrow \mathbb{H}P^n$ and $S^8 \hookrightarrow \mathbb{O}P^2$ together with the knowledge of the BV operator on $\mathbb{H}_*(S^n;\mathbb{Z})$, see [9].

We start with the quaternionic projective space. First, $\Delta(a^px^q) = 0$ because $\mathbb{H}_{|a^px^q|+1}(L\mathbb{H}P^n;\mathbb{Z}) = 0$. Since $\mathbb{H}_{|a^pb|+1}(L\mathbb{H}P^n;\mathbb{Z}) \cong \mathbb{Z}$ is generated by a^p, there is an integer ν_p such that $\Delta(a^pb) = \nu_p a^p$. Due to the relation

$$\begin{aligned}\Delta(xyz) =& \Delta(xy)z + (-1)^{|x|}x\Delta(yz) + (-1)^{(|x|-1)|y|}y\Delta(xz) \\ & - \Delta(x)yz - (-1)^{|x|}x\Delta(y)z - (-1)^{|x|+|y|}xy\Delta(z)\end{aligned}$$

we obtain

$$\Delta(a^pb) = \Delta(a^{p-1}ab) = a^{p-1}\Delta(ab) + a\Delta(a^{p-1}b) - a^p\Delta(b).$$

It yields the equation $\nu_p = \nu_1 - \nu_0 + \nu_{p-1}$, which can be rewritten as

$$\nu_p = p(\nu_1 - \nu_0) + \nu_0.$$

The relation $a^nb = 0$ implies that $\nu_n = 0$. Consequently, for $p = n$ the equation above gives $n\nu_1 = (n-1)\nu_0$. Hence for $n \geq 2$ the only possible integer solutions of this equation are

$$\nu_1 = (n-1)\lambda_n, \quad \nu_0 = n\lambda_n,$$

where λ_n is an integer. Consequently, we obtain $\nu_p = (n-p)\lambda_n$.

For $n = 1$ the quaternionic projective space is S^4. According to [9] the generators of $\mathbb{H}_*(L\mathbb{H}P^1;\mathbb{Z})$ as an algebra are a_1, b_1 and v_1 in degrees -4, -1 and 6, respectively, and $\Delta(b_1) = 1$.

The standard inclusion $i : S^4 = \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^n$ induces the commutative diagram of fibrations

$$\begin{tikzcd}
\Omega\mathbb{H}P^1 \arrow[r] \arrow[d] & \Omega\mathbb{H}P^n \arrow[d] \\
L\mathbb{H}P^1 \arrow[r] \arrow[d] & L\mathbb{H}P^n \arrow[d] \\
\mathbb{H}P^1 \arrow[r] & \mathbb{H}P^n
\end{tikzcd}$$

The inclusion i induces an isomorhism $H_4(\mathbb{H}P^1;\mathbb{Z}) \cong H_4(\mathbb{H}P^n;\mathbb{Z})$ and the inclusion $\Omega\mathbb{H}P^1 \hookrightarrow \Omega\mathbb{H}P^n$ yields an isomorphism $H_3(\Omega\mathbb{H}P^1;\mathbb{Z}) \cong H_3(\Omega\mathbb{H}P^n;\mathbb{Z})$.

The commutative diagram above gives us a homomorphism between the Serre spectral sequences of the corresponding fibrations. (Here we consider the spectral sequences with $E^2_{p,q} = H_p(M; H_q(\Omega M;\mathbb{Z}))$.) This homomorphism is an isomorphism on $E^2_{4,0}$ and $E^2_{0,3}$ terms and it remains an isomorphism also on $E^\infty_{4,0}$ and $E^\infty_{0,3}$. Consequently, for $i = -1$ and 0 we obtain

$$\mathbb{H}_i(LS^4;\mathbb{Z}) = H_{i+4}(LS^4;\mathbb{Z}) \cong H_{i+4}(L\mathbb{H}P^n;\mathbb{Z}) = \mathbb{H}_{i-4(n-1)}(L\mathbb{H}P^n;\mathbb{Z}).$$

Choose $b \in \mathbb{H}_1(L\mathbb{H}P^n;\mathbb{Z})$ and $a \in \mathbb{H}_4(L\mathbb{H}P^n;\mathbb{Z})$ so that a^{n-1} is the image of 1 and $a^{n-1}b$ is the image of $b_1 \in \mathbb{H}_1(LS^4;\mathbb{Z})$ under the above isomorphisms. Since these isomorphisms commute with Δ, we obtain $\Delta(a^{n-1}b) = a^{n-1}$. Consequently, $\lambda_n = 1$.

Analogously we get $\Delta(bx^q) = \rho_q x^q$ for an integer ρ_q and derive that

$$\rho_q = q(\rho_1 - \rho_0) + \rho_0.$$

Since $\rho_0 = \nu_0 = n\lambda_n = n$, we obtain

$$\begin{aligned}\Delta(a^p b x^q) &= \Delta(a^p b)x^q + a^p\Delta(bx^q) - a^p x^q \Delta(b) = \\ &= [(n-p) + q(\rho_1 - n)]a^p x^q.\end{aligned}$$

In [13] T. Yang computed the BV-algebra structure of the Hochshild cohomology of truncated polynomials. Using Theorem 1 from [5] on the existence of a BV-algebra isomorhism between the loop homology $\mathbb{H}_*(LM;\mathbb{F})$ of a manifold and the Hochschild cohomology $HH^*(C^*(M);C^*(M))$ of the singular cochain complex over fields of characteristic zero, he was able to calculate the BV-algebra structure of $\mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Q})$. This is given by

$$\mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Q}) = \frac{\mathbb{Q}[\alpha,\beta,\chi]}{\langle \alpha^{n+1}, \beta^2, \alpha^n\beta, \alpha^n\chi\rangle},$$

where $|\alpha| = -4$, $|\beta| = -1$, $|\chi| = 4n+2$, and by

$$\Delta(\alpha^p\chi^q) = 0, \quad \Delta(\alpha^p\beta\chi^q) = [(n-p) + q(n+1)]\alpha^p\chi^q.$$

Consider the homomorphism $r_* : \mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Z}) \to \mathbb{H}_*(L\mathbb{H}P^n;\mathbb{Q})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Let

$$r_*(a) = k\alpha, \quad r_*(b) = l\beta, \quad r_*(x) = m\chi,$$

where, $k, l, m \in \mathbb{Q} - \{0\}$. Since r_* is a homomorphism of BV-algebras, we obtain

$$[(n-p)+q(\rho_1-n)]k^p m^q \alpha^p\chi^q = r_*(\Delta(a^p b x^q) = \Delta(r_*(a^p b x^q)) = l[(n-p)+q(n+1)]k^p m^q \alpha^p\chi^q.$$

Putting $q = 0$ we get $l = 1$. Then the choice $p = 0$, $q = 1$ yields $\rho_1 = 2n + 1$ which concludes our computation.

To compute the BV operator in $\mathbb{H}_*(L\mathbb{O}P^2; \mathbb{Z})$ we can follow the same procedure step by step replacing the inclusion $S^4 \hookrightarrow \mathbb{H}P^n$ by the inclusion $S^8 \hookrightarrow \mathbb{O}P^2$.

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